CENTER FOR BEAM PHYSICS SEMINAR

“Historical Roots of Gauge Invariance”

J. D. Jackson (UCB/LBNL)

Friday January 11, 2002, 10:30 AM
Albert Ghiorso Conference Room (71-264), LBNL

Abstract: A number of reviews of gauge theories cover the period from about 1929 (Weyl's major paper on the subject) to the present day, with stress on the post-Yang-Mills epoch. Lev Okun and I address the "pre-history" of the subject, starting with Ampère, Neumann, Weber, and others, and the debates over the "correct" form of the vector potential. The story continues with Maxwell, Lorenz, Helmholtz, Clausius, and Lorentz by which time the idea of different, equivalent gauges for the potentials in classical electromagnetism had been clarified completely. We then discuss the *annus mirabilus*, 1926, with Fock's discovery of the phase transformation of the wave function that must accompany a gauge change of the potentials. The unfair belittlement of the contributions of Lorenz and Fock are aired. Portraits of all the "electricians" will be presented as the story unfolds. [Reference: J. D. Jackson and Lev Okun, Rev. Mod. Phys. Vol. 73, 663-680 (2001)]

Biographical data: Dave Jackson is a Professor Emeritus of Physics, UC Berkeley, and a Senior Physicist at LBNL since 1967. He is the author of a well-known text, *Classical Electrodynamics*. His research interests are mainly in particle physics, but in his retirement he has been writing pedagogical articles in the American Journal of Physics as well as the recent foray into the history of science.

CPB SEMINAR, LBNL
JANUARY 11, 2002

J. D. JACKSON

HISTORICAL ROOTS OF GAUGE INVARIANCE

[PAPER WITH LEV OKUN, JULY 2001 RMP]

WHAT IS GAUGE INVARIANCE ?

(a) CLASSICAL E & M

FIELDS $(\vec{E}, \vec{B})$, POTENTIALS $(\Phi, \vec{A})$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

POTENTIALS NOT UNIQUE:

$$\Phi' = \Phi - \frac{1}{c}\frac{\partial \chi}{\partial t} \; ; \; \vec{A}' = \vec{A} + \vec{\nabla}\chi$$

$(\Phi', \vec{A}')$ GIVE SAME FIELDS AS $(\Phi, \vec{A})$

WE CALL THIS "INVARIANCE" OF THE FIELDS AND THE MAXWELL EQUATIONS OR THE LAGRANGIAN, "GAUGE INVARIANCE"

(b) QUANTUM MECHANICS

SCHRÖDINGER EQUATION FOR CHARGED PARTICLE

$$\left[\frac{1}{2m}\left(\vec{P}-e\vec{A}/c\right)^2+e\Phi\right]\psi=i\hbar\frac{\partial\psi}{\partial t}$$

IF $(\Phi,\vec{A})\to(\Phi',\vec{A}')$, WE MUST HAVE

$$\psi\to\psi'=\psi e^{ie\chi/\hbar c}$$

[DISCOVERED BY VLADIMIR FOCK, 1926]

(c) QFT AND GAUGE PRINCIPLE

WEYL (1929) ENSHRINED GAUGE INVARIANCE AS A SYMMETRY PRINCIPLE BY RUNNING THINGS BACKWARDS:

IF WAVE FUNCTION OR FIELD HAS A LOCAL SYMMETRY, $\psi\to\psi'=\psi e^{i\chi(x)}$

WHERE $\chi(x)$ IS FUNCTION OF (x,y,z,t), THEN THE THEORY "REQUIRES" THE INTRODUCTION OF VECTOR FIELDS, A^μ, COUPLED IN A UNIQUE WAY TO MATTER

SINCE 1954 (YANG + MILLS), QUANTUM GAUGE FIELDS HAVE BECOME A HUGE INDUSTRY (QED, QCD, ELECTROWEAK, ...)

REVIEW:

L. O'RAIFEARTAIGH + N. STRAUMANN, REV. MOD. PHYS. 72, 1–23 (JAN 2000)

OTHERS BY OKUN (1986), YANG (1986, 1987), O'R (1997)

OUR TOPIC IS THE "PRE-HISTORY":

CLASSICAL (1820 → 1900+)

QUANTUM (1926 → 1929)

BUT FIRST, A PLUG FOR

U.C. BERKELEY'S DOE LIBRARY

(AND PHYSICS LIBRARY)

100 – 175 YEAR OLD BOOKS AND JOURNALS AVAILABLE!

ALSO BANCROFT LIBRARY

+ INTER-LIBRARY LOAN

MÉMOIRES

DE

L'ACADÉMIE ROYALE DES SCIENCES DE L'INSTITUT DE FRANCE.

ANNÉE 1823.

TOME VI.

PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS, LIBRAIRES,
RUE JACOB, N° 24.

1827.

PAPERS BY

LEGENDRE
LAPLACE
AMPÈRE

NAVIER
POISSON
CAUCHY
FOURIER

FIRST PAGE OF AMPÈRE'S MEMOIR

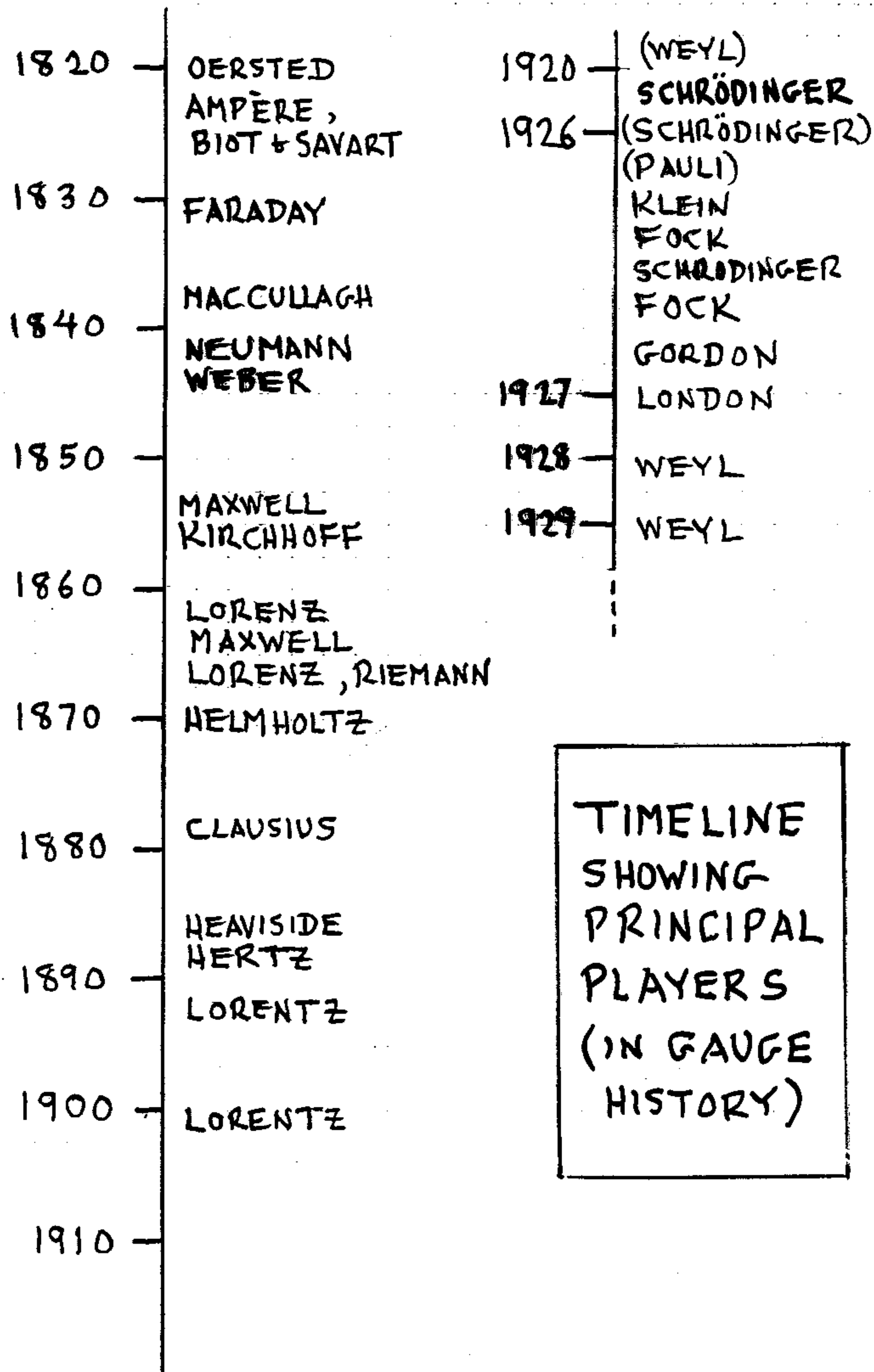
MÉMOIRE

Sur la théorie mathématique des phénomènes électro-dynamiques uniquement déduite de l'expérience, dans lequel se trouvent réunis les Mémoires que M. Ampère a communiqués à l'Académie royale des Sciences, dans les séances des 4 et 26 décembre 1820, 10 *juin* 1822, 22 *décembre* 1823, 12 *septembre et* 21 *novembre* 1825.

L'ÉPOQUE que les travaux de Newton ont marquée dans l'histoire des sciences n'est pas seulement celle de la plus importante des découvertes que l'homme ait faites sur les causes des grands phénomènes de la nature, c'est aussi l'époque où l'esprit humain s'est ouvert une nouvelle route dans les sciences qui ont pour objet l'étude de ces phénomènes.

Jusqu'alors on en avait presque exclusivement cherché les causes dans l'impulsion d'un fluide inconnu qui entraînait les particules matérielles suivant la direction de ses propres particules; et partout où l'on voyait un mouvement révolutif, on imaginait un tourbillon dans le même sens.

Newton nous a appris que cette sorte de mouvement doit, comme tous ceux que nous offre la nature, être ramenée par le calcul à des forces agissant toujours entre deux particules matérielles suivant la droite qui les joint, de manière que



TIMELINE SHOWING PRINCIPAL PLAYERS (IN GAUGE HISTORY)

André - Marie Ampère
(1775 - 1836)

OERSTED (JULY, 1820)

AMPÈRE IMMEDIATELY LEAPT INTO ACTION - INTERACTION BETWEEN CLOSED CIRCUITS, ETC.

[FIG. 1 OF OUR PAPER]

ROOT OF CONFUSION → DIFFERENT VECTOR POTENTIALS → GAUGE INV.

TOOK 80 YEARS TO CLARIFY

CAUSE OF CONFUSION?

AMPÈRE'S DESIRE FOR DIFFERENTIAL EXPRESSION FOR FORCE (ASSUMED TO BE CENTRAL)

$$dF = 4k\frac{II'}{\sqrt{r}}\frac{\partial^2\sqrt{r}}{\partial s\,\partial s'}\,ds\,ds'$$

IN MORE FAMILIAR NOTATION,

$$d\mathbf{F} = \frac{II'}{c^2}\frac{\hat{\mathbf{r}}}{r^2}\left[\,3\,\hat{\mathbf{r}}\cdot\mathbf{n}\,\hat{\mathbf{r}}\cdot\mathbf{n}' - 2\,\mathbf{n}\cdot\mathbf{n}'\,\right]ds\,ds'$$

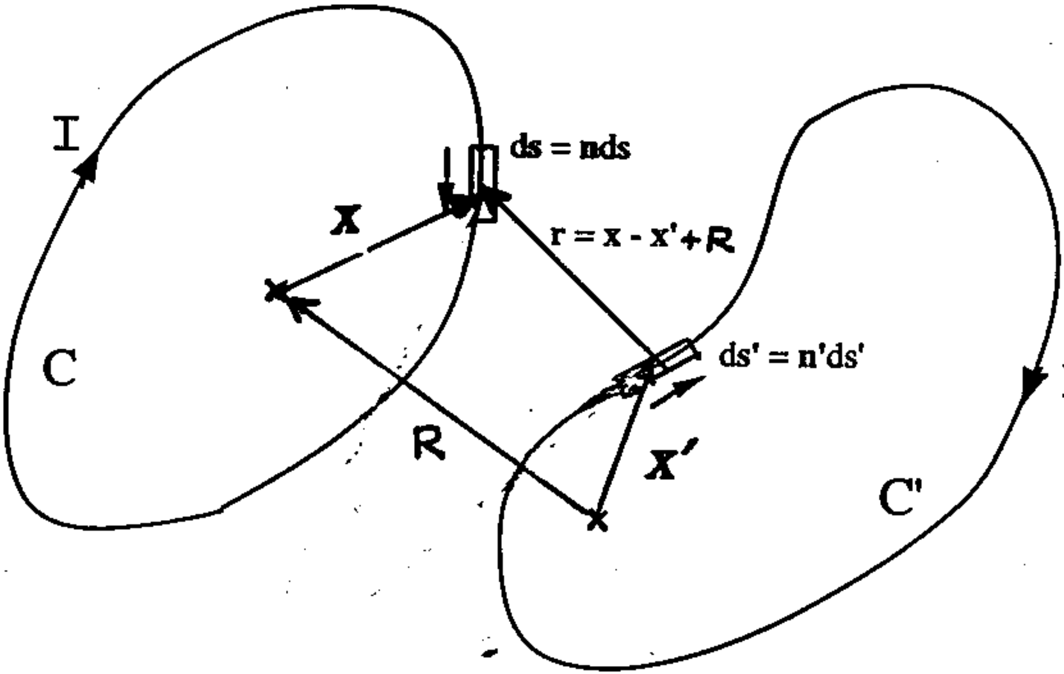


FIGURE 1, JACKSON & OKUN

AMPÈRE'S <u>EXPRESSION</u> <u>IS</u> "<u>WRONG</u>";
<u>YET</u> HE IS CREDITED WITH MAKING THE <u>RIGHT</u> <u>DISCOVERY</u> OF FORCES BETWEEN CURRENTS!

<u>HOW CAN THAT BE</u>?

PRESENT-DAY DIFFERENTIAL FORCE IS

$$d\mathbf{F} = \frac{II'}{c^2 r^2}\mathbf{n} \times (\mathbf{n}' \times \hat{\mathbf{r}})\, ds\, ds'$$

$$= \frac{II'}{c^2 r^2}\left[\mathbf{n}'(\hat{\mathbf{r}}\cdot\mathbf{n}) - \hat{\mathbf{r}}(\mathbf{n}\cdot\mathbf{n}')\right] ds\, ds',$$

(NEUMANN, GRASSMANN, 1845)

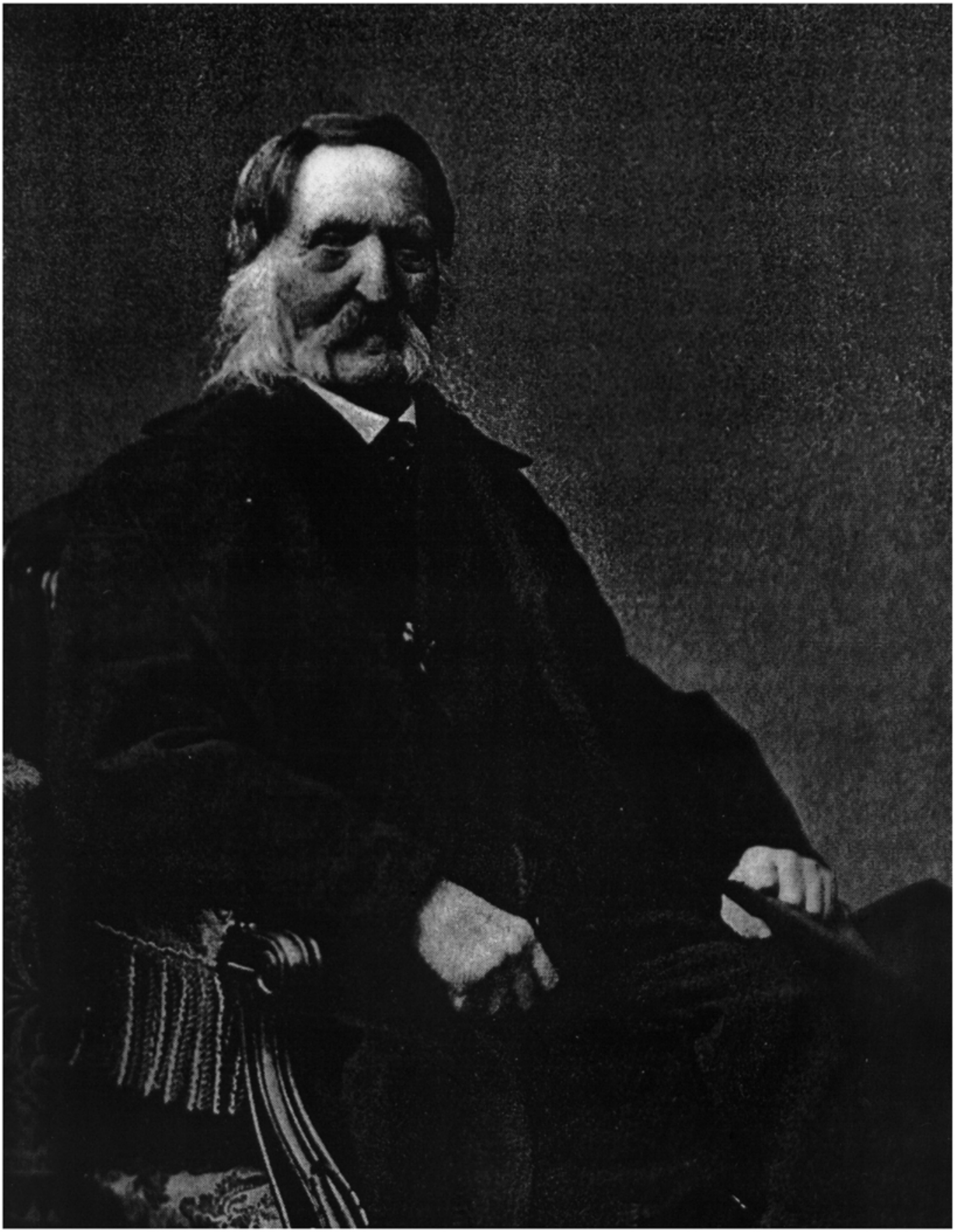
THE DIFFERENCE IS

$$d\vec{F}_A - d\vec{F} = \frac{II'}{c^2 r^2}\left[3\hat{r}(\hat{r}\cdot\vec{n})(\hat{r}\cdot\vec{n}') - \hat{r}(\vec{n}\cdot\vec{n}') - \vec{n}'(\hat{r}\cdot\vec{n})\right] ds\, ds'$$

THIS CAN BE WRITTEN AS

$$d\vec{F}_A - d\vec{F} = \frac{II'}{c^2}\left[d\vec{s}\, d\vec{s}'\cdot\vec{\nabla}'\left(\frac{1}{r}\right) - \vec{\nabla}_R \frac{\partial^2 r}{\partial s\, \partial s'}\, ds\, ds'\right]$$

<u>PRESENCE</u> OF <u>PERFECT</u> <u>DIFFERENTIALS</u> MEANS <u>DIFFERENCE</u> <u>VANISHES</u> <u>UPON</u> <u>INTEGRATION</u> <u>OVER</u> <u>CLOSED</u> <u>CIRCUITS</u>!



Franz E. Neumann
(1798 - 1895)

COMPETING VERSIONS OF DIFFERENTIAL FORCE LED TO COMPETING EXPRESSIONS FOR ENERGY

1. **NEUMANN** :

DROPPED PERFECT DIFFERENTIAL FROM HIS FORCE, NOTED THAT $-\vec{\nabla}_R\left(\frac{1}{r}\right) = \frac{\hat{r}}{r^2}$,

AND DEFINED "MAGNETIC POTENTIAL ENERGY"

$$dP = -\frac{II'}{c^2}\frac{\mathbf{n}\cdot\mathbf{n}'}{r}\,ds\,ds' \;; \quad P = -\frac{II'}{c^2}\oint_C\oint_{C'}\frac{\mathbf{n}\cdot\mathbf{n}'}{r}\,ds\,ds'$$

NOWADAYS, WE DEFINE $W = -P$. IN TERMS OF CURRENT DENSITIES $J(\vec{x},t), J'(\vec{x}',t)$

$$W = \frac{1}{c^2}\int d^3x\int d^3x'\,\frac{\vec{J}'\cdot\vec{J}}{r} = \frac{1}{c}\int d^3x'\,\vec{J}'\cdot\vec{A}_N$$

WHERE NEUMANN'S VECTOR POTENTIAL IS

$$\boxed{\mathbf{A}_N(\mathbf{x},t) = \frac{1}{c}\int d^3x'\,\frac{1}{r}\,\mathbf{J}(\mathbf{x}',t)}$$

Wilhelm E. Weber
(1804 - 1891)

2. WEBER :

WANTED TO UNITE ELECTROMAGNETICS AND ELECTROSTATICS.

CURRENT FLOW WAS NOT FLUID FLOW, BUT CHARGED PARTICLES IN MOTION :

$$\text{CURRENT}\left[I \longrightarrow \right] = \left[\begin{array}{c} + \longrightarrow v \\ - \longleftarrow -v \end{array} \right]$$

$I = Nev$ (CONFUSING FACTORS OF 2 !

WEBER INVENTED A BASIC FORCE EQUATION FOR HIS CHARGES:

$$F = \underbrace{\frac{ee'}{r^2}}_{\text{COULOMB FORCE}} + \frac{ee'}{c^2}\underbrace{\left[\frac{1}{r}\frac{d^2r}{dt^2} - \frac{1}{2r^2}\left(\frac{dr}{dt}\right)^2\right]}_{\text{FORCE BETWEEN CIRCUITS}}$$

COOKED UP TO GIVE AGREEMENT WITH
(i) AMPÈRE'S EXPERIMENTS
(ii) FARADAY'S LAW OF INDUCTION

FOR (ii), WEBER'S FORCE ON A STATIONARY CHARGE e is

$$dF = -\frac{e}{c^2 r}\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{n}'\frac{dI'}{dt}ds'$$

IF THE CURRENT IN CIRCUIT C' CHANGES IN TIME

IF WE IDENTIFY $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, WE GET

$$d\vec{A} = \frac{I'}{cr}\hat{r}(\hat{r}\cdot\vec{m}')\,ds'$$

FOR A DISTRIBUTION OF CURRENT DENSITY THIS GIVES THE WEBER FORM OF THE VECTOR POTENTIAL,

$$\boxed{\mathbf{A}_W(\mathbf{x}, t) = \frac{1}{c}\int d^3x'\,\frac{1}{r}\,\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{J}(\mathbf{x}', t)}$$

NEUMANN AND WEBER NEVER WROTE $\vec{A}_N$ OR $\vec{A}_W$ SEPARATELY. THEY ALWAYS WROTE FORMULAS FOR TWO CIRCUITS INTERACTING.

KIRCHHOFF (1857) WAS THE FIRST TO WRITE OUT $\vec{A}_W$ (IN COMPONENT FORM) IN A PAPER ANALYZING THE TELEGRAPH.

HE ESTABLISHED THE RELATION,

$$\vec{\nabla}\cdot\vec{A}_W = \frac{\partial \Phi}{c\,\partial t}$$

FOR THE QUASI-STATIC (INSTANTANEOUS) VECTOR AND SCALAR POTENTIALS

— FIRST PUBLISHED RELATION BETWEEN THE POTENTIALS (IN A PARTICULAR GAUGE) (NOT LORENZ CONDITION)

Gustav Kirchhoff
(1824 - 1897)

Hermann von Helmholtz
(1821 - 1894)

WHO IS CORRECT, NEUMANN OR WEBER?

HELMHOLTZ OBJECTED TO WEBER'S FORCE AS UNPHYSICAL — THE PRESENCE OF THE ACCELERATION WAS PARTICULARLY BOTHERSOME

BUT HELMHOLTZ KNEW THAT WEBER'S FORM FOR $\vec{A}$ WORKED. IN 1870-74 HELMHOLTZ SOLVED THE PUZZLE:

$$dW = \frac{II'}{c^2 r} ds\, ds' \begin{Bmatrix} \vec{m}\cdot\vec{m}' \\ \hat{r}\cdot\vec{m}\;\hat{r}\cdot m' \end{Bmatrix} \begin{matrix} \text{NEUMANN} \\ \text{WEBER} \end{matrix}$$

HELMHOLTZ NOTED THAT THE TWO FORMS DIFFERED BY A PERFECT DIFFERENTIAL,

$$ds\, ds' \frac{\partial^2 r}{\partial s\, \partial s'} = \frac{ds\, ds'}{r}\left(\hat{r}\cdot\vec{m}\;\hat{r}\cdot\vec{m}' - \vec{m}\cdot\vec{m}'\right)$$

[WE SAW THE GRADIENT OF THIS BEFORE!]

THE TOTAL ENERGIES ARE THE SAME!

HELMHOLTZ CONCLUDED THAT THE VECTOR POTENTIALS WERE EQUIVALENT AND GENERALIZED:

$$\boxed{A_\alpha = \frac{1}{2}(1+\alpha)\, A_N + \frac{1}{2}(1-\alpha)\, A_W}$$

FIRST EXAMPLE OF A (RESTRICTED) CLASS OF DIFFERENT GAUGES.

HELMHOLTZ WENT ON TO FIND THE GAUGE FUNCTION χ. HE SHOWED THAT

$$\mathbf{A}_\alpha = \mathbf{A}_N + \frac{(1-\alpha)}{2}\nabla\Psi\ ,$$

where $\Psi = -\frac{1}{c}\int \hat{\mathbf{r}}\cdot\mathbf{J}(\mathbf{x}', t)\, d^3x'$

AND THAT

$$\boxed{\nabla\cdot\mathbf{A}_\alpha = -\alpha\frac{\partial\Phi}{c\,\partial t}}$$

$\alpha=-1$ GIVES KIRCHHOFF'S RELATION FOR $\vec{A}_W$

$\alpha=+1$ GIVES QUASI-STATIC LORENZ RELATION

HELMHOLTZ SAYS (NOT QUITE CORRECTLY) THAT $\alpha=0$ LEADS TO MAXWELL'S THEORY

FOR $\alpha=0$, THE EQUAL PARTS OF $\vec{A}_N$ AND $\vec{A}_W$ IS

$$\mathbf{A}_M(\mathbf{x},t) = \frac{1}{2c}\int\left(\frac{\mathbf{J}(\mathbf{x}',t)}{r}+\frac{\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{J}(\mathbf{x}',t)}{r}\right)d^3x'$$

MAXWELL NEVER WROTE THIS EXPRESSION, BUT $\alpha=0$ IMPLIES $\vec{\nabla}\cdot\vec{A}_M=0$

AND THE CHOICE $\vec{\nabla}\cdot\vec{A}=0$ WAS ALWAYS MADE BY MAXWELL.

JAMES CLERK MAXWELL
(1831 - 1879)

IN THE 1850s MAXWELL DEVELOPED HIS OWN APPROACH TO QUANTIFY FARADAY'S INTUITIVE IDEA OF A CIRCUIT IMMERSED IN A MAGNETIC FIELD BEING IN AN "ELECTRO-TONIC STATE," READY TO RESPOND WITH CURRENT FLOW IF THE MAGNETIC FLUX THROUGH IT CHANGED.

AWARE OF WEBER'S APPROACH, HE AVOIDED CHARGED PARTICLES IN MOTION AND GOT TO EMFs AND $c\vec{E} = -\frac{d\vec{A}}{dt} = -\frac{\partial \vec{A}}{\partial t} - (\vec{v}\cdot\vec{\nabla})\vec{A}$

WITH $\vec{A}$ BEING THE NEUMANN FORM.

HE CALLED $\vec{A}$ "ELECTROMAGNETIC MOMENTUM" CITING THE ANALOGY WITH $\vec{F} = \frac{d\vec{p}}{dt}$

ALSO "ELECTROKINETIC MOMENTUM" AS WELL AS "VECTOR POTENTIAL". [ALL THREE IN HIS TREATISE (1873)]

HIS CHOICE OF $\vec{\nabla}\cdot\vec{A} = 0$, WITH ITS CONSEQUENCE OF AN INSTANTANEOUS SCALAR POTENTIAL, WAS A RESULT OF HIS DESIRED PARALLELISM,

$$-\nabla^2 \Phi = 4\pi\rho$$

$$-\nabla^2 \vec{A} = \frac{4\pi}{c}\vec{J}_{total} = \frac{4\pi}{c}\left(\vec{J} + \frac{1}{4\pi}\frac{\partial \vec{D}}{\partial t}\right)$$

HE LIKED THE IDEA OF $\vec{\nabla}\cdot\vec{J}_{total} = 0$.

|| DISCONNECT BETWEEN INSTANTANEOUS Φ AND PROPAGATION WITH FINITE SPEED !

Ludvig Valentin Lorenz
(1829 - 1891)

LUDVIG VALENTIN LORENZ (DANISH)

1863 "ON THE THEORY OF LIGHT"
PHENOMENOLOGICAL – TRANSVERSE VIBRATIONS, ETC.

1867 INDEPENDENT OF MAXWELL (1865)

AVOIDS DISTASTEFUL AETHER HYPOTHESIS;

FOLLOWS HELMHOLTZ AND ASSUMES ALL MEDIA (INCL. "EMPTY" SPACE) CONDUCT $\vec{J} = \sigma \vec{E}$; STARTING POINT:

(1) INSTANTANEOUS Φ AND $\vec{A}_W$ (KIRCHHOFF), BUT

(2) ASSERTS FINITE SPEED OF PROPAGATION

(3) ASSUMES RETARDED Φ AND $\vec{A}$:

$$\Phi(\mathbf{x}, t) = \int \frac{\rho(\mathbf{x}', t - r/c)}{r} d^3x' ;$$

NEUMANN-LIKE "FOR SIMPLICITY" (SINCE IT DOESN'T MATTER)

AND

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{x}', t - r/c)}{r} d^3x'$$

ARGUES THAT

ALL STATIC AND QUASI-STATIC OBSERVATIONS ARE CONSISTENT WITH THESE FORMS

(4) HE HAD SHOWN (1861, PAPER ON ELASTIC WAVES) THAT THESE RETARDED SOLUTIONS SATISFY THE WAVE EQUATION,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left\{ \begin{matrix} \Phi \\ \vec{A} \end{matrix} \right\} = \left\{ \begin{matrix} 4\pi\rho \\ \frac{4\pi}{c}\vec{J} \end{matrix} \right\}$$

(5) APPLY WAVE OPERATOR TO $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

TO GET $$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{E} = -4\pi\left(\vec{\nabla}\rho + \frac{1}{c^2}\frac{\partial \vec{J}}{\partial t}\right)$$

WHICH CLEARLY HAS WAVE-LIKE SOLUTIONS FOR $\vec{E}$

(6) USES $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ IN THE RETARDED

SOLUTIONS FOR Φ AND $\vec{A}$ TO DERIVE

25+ YEARS BEFORE LORENTZ!

$$\boxed{\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0} \qquad \frac{d\Omega}{dt} = -2\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right).$$

(7) WRITES $$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} = -\frac{1}{c}\vec{\nabla}\frac{\partial \Phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$

$$= \vec{\nabla}\times\vec{\nabla}\times\vec{A} + \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}$$

FINALLY $$\boxed{\frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})}$$ {AMPÈRE-MAXWELL}

$$\left.\begin{aligned}
\frac{1}{4k}\frac{du}{dt} + 4\pi u &= \frac{d}{dz}\left(\frac{d\gamma}{dx} - \frac{d\alpha}{dz}\right) - \frac{d}{dy}\left(\frac{d\alpha}{dy} - \frac{d\beta}{dx}\right), \\
\frac{1}{4k}\frac{dv}{dt} + 4\pi v &= \frac{d}{dx}\left(\frac{d\alpha}{dy} - \frac{d\beta}{dx}\right) - \frac{d}{dz}\left(\frac{d\beta}{dz} - \frac{d\gamma}{dy}\right), \\
\frac{1}{4k}\frac{dw}{dt} + 4\pi w &= \frac{d}{dy}\left(\frac{d\beta}{dz} - \frac{d\gamma}{dy}\right) - \frac{d}{dx}\left(\frac{d\gamma}{dx} - \frac{d\alpha}{dz}\right).
\end{aligned}\right\} \quad (8)$$

$$\left(\frac{1}{4k}u = \frac{\vec{J}}{\sigma c} = \frac{\vec{E}}{c}\right)$$

(8) GETS DIRECTLY FROM $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})}$$ {FARADAY'S LAW}

(9) CHOOSES TO ELIMINATE $\vec{A}$ IN FAVOR OF $\vec{E}$ (ACTUALLY $\vec{J}$) TO GET

$$\underline{-\vec{\nabla}\times(\vec{\nabla}\times\vec{E})} = \underline{\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}} + \underbrace{\frac{4\pi\sigma}{c^2}\frac{\partial \vec{E}}{\partial t}}_{\text{(OHM'S LAW CONDUCTION)}}$$

LORENZ DISCUSSES

DIELECTRICS $(\sigma \to 0^+)$

METALS (LARGE σ) – NO FREE CHARGE INSIDE

LORENZ SHOULD RANK WITH MAXWELL!

MUCH MORE MODERN VIEWPOINT

BUT IN 1868, MAXWELL WROTE ABOUT THE RETARDED SOLUTIONS FOR Φ AND $\vec{A}$:

"From the assumptions of both these papers we may draw the conclusions, first, that action and reaction are not always equal and opposite, and second, that apparatus may be constructed to generate any amount of work from its resources." (Maxwell, 1868, *Sc. P.*, Vol. 2, p. 137).

KISS OF DEATH! CONSERVATION OF $\vec{P}$, E SACROSANCT + MAXWELL!

IRONIC – ELECTROMAGNETIC MOMENTUM!

Rudolf J. E. Clausius
(1822 - 1888)

LORENZ'S WORK WAS LARGELY IGNORED. BY 1900, HE HAD DISAPPEARED FROM THE LITERATURE.

CHARGED PARTICLE DYNAMICS

WEBER - NOT A PROPER FORCE

CLAUSIUS (1877)

TOOK $dW = \frac{II'}{c^2 r} \vec{m} \cdot \vec{m}' ds\, ds'$ (NEUMANN)

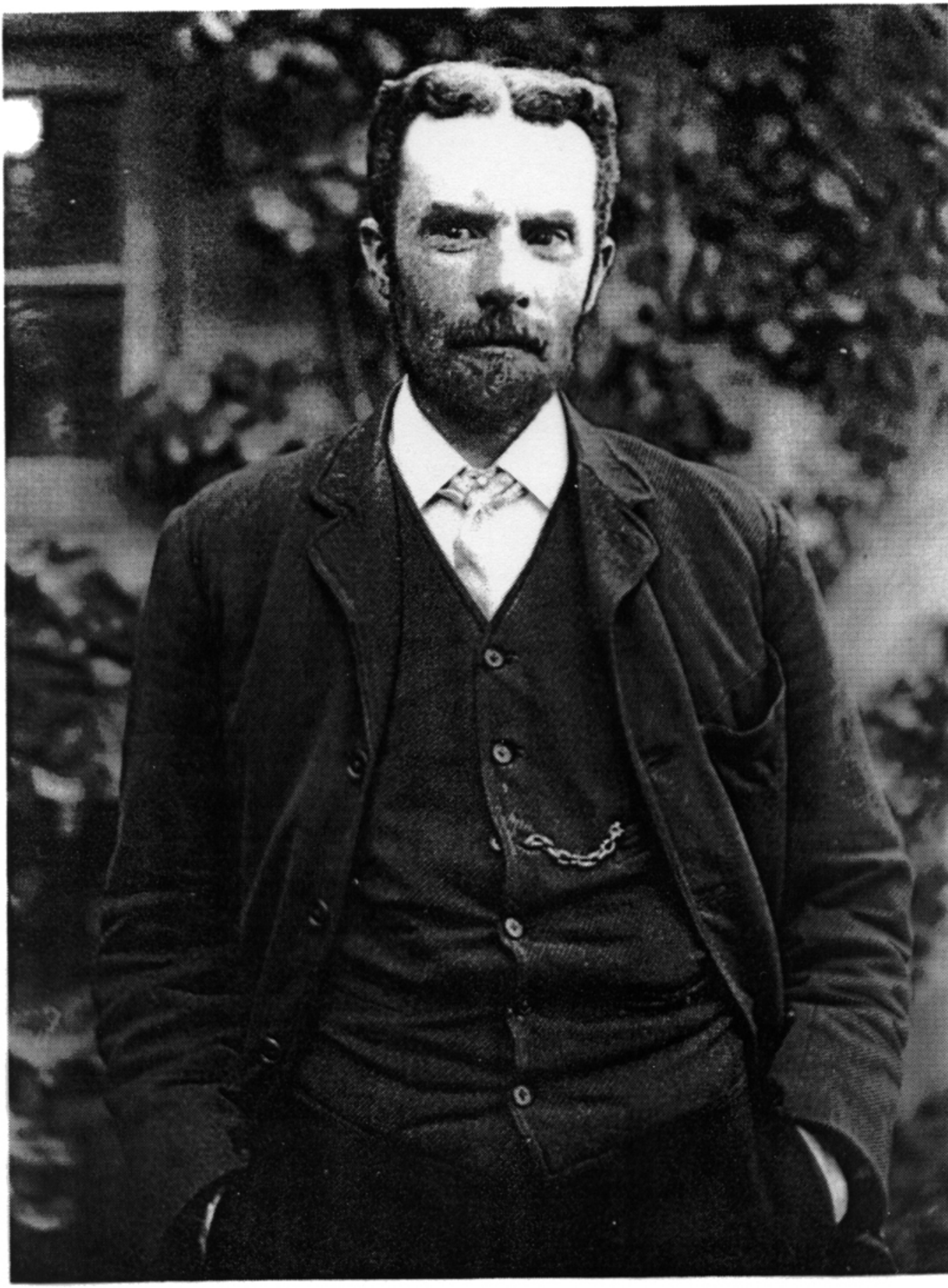
AND IDENTIFIED $I\vec{m}\,ds \to e\vec{v}$, $I'\vec{m}'ds' \to e'\vec{v}'$ TO GET INTERACTION LAGRANGIAN,

$$L_{int} = \frac{ee'}{r}\left[-1 + \frac{\mathbf{v}\cdot\mathbf{v}'}{c^2}\right]$$ FOR TWO CHARGES

OR FOR A PARTICLE IN EXTERNAL POTENTIALS

$$L_{int} = e\left[-\Phi(\mathbf{x}, t) + \frac{1}{c}\mathbf{v}\cdot\mathbf{A}_N(\mathbf{x}, t)\right]$$

IMPROVEMENT OVER WEBER - GETS MAGNETIC FORCE CORRECT TO $O(1/c^2)$, BUT NOT ELECTRIC FORCE.



Oliver Heaviside
(1850 - 1925)

HEAVISIDE (1889)

CHOSE $\vec{\nabla}\cdot\vec{A}=0$ SO THAT INSTANTANEOUS COULOMB INTERACTION WAS EXACT, AND DERIVED THE VELOCITY-DEPENDENT INTERACTION TO ORDER v^2/c^2:

$$\boxed{L_{int} = \frac{e\,e'}{r}\left[-1+\frac{1}{2c^2}\left(\mathbf{v}\cdot\mathbf{v}'+\hat{\mathbf{r}}\cdot\mathbf{v}\,\hat{\mathbf{r}}\cdot\mathbf{v}'\right)\right]}$$

[RECALL $W=\frac{1}{c}\int\vec{J}_1\cdot\vec{A}_2\,d^3x$ AND "MAXWELL'S"

$$A_M(\mathbf{x},t) = \frac{1}{2c}\int\left(\frac{\mathbf{J}(\mathbf{x}',t)}{r}+\frac{\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{J}(\mathbf{x}',t)}{r}\right)d^3x'$$]

THIS IS THE "DARWIN LAGRANGIAN" (1920)

LORENTZ (1892, 1895)

USES D'ALEMBERT'S PRINCIPLE TO DERIVE WHAT WE CALL "MICROSCOPIC" MAXWELL EQUATIONS AND LORENTZ FORCE EQUATION

IMPLIED USE OF RETARDED POTENTIALS, BUT NEVER WROTE THEM OUT EXPLICITLY.

IN LATER CHAPTER, HE WROTE RETARDED SOLUTION TO WAVE EQUATION WITHOUT REF. TO LORENZ OR RIEMANN

Hendrik Antoon Lorentz
(1853 - 1928)

LARMOR (1900), SCHWARZSCHILD (1903)

FIRST USE OF PRINCIPLE OF LEAST ACTION TO DERIVE DYNAMICS OF FIELDS AND PARTICLES TOGETHER. SCHWARZSCHILD WRITES EXPLICITLY,

$$L_{INT} = e\left[-\Phi_{RET}(\vec{x},t) + \frac{\vec{v}}{c}\cdot\vec{A}_{RET}(\vec{x},t)\right]$$

WHERE Φ_{RET} AN $\vec{A}_{RET}$ ARE POTENTIALS FROM ALL THE OTHER CHARGED PARTICLES

GAUGE INVARIANCE ?

NO PUBLISHED DISCUSSION UNTIL 1941 (LANDAU + LIFSHITZ, "CTF" (IN RUSSIAN))

WITH $\Phi \to \Phi - \frac{1}{c}\frac{\partial\chi}{\partial t}$, $\vec{A} \to \vec{A} + \vec{\nabla}\chi$,

$$\Delta L_{INT} = \frac{e}{c}\left[\frac{\partial\chi}{\partial t} + \vec{v}\cdot\vec{\nabla}\chi\right] = \frac{e}{c}\frac{d\chi}{dt},$$

A TOTAL DERIVATIVE

BY 1900, FESTSCHRIFT FOR LORENTZ

PAEANS OF PRAISE FOR H.A. LORENTZ, "HISTORICAL" ACCOUNTS CITE HELMHOLTZ, RIEMANN, POINCARÉ, ~~ETC~~. BUT NEVER LORENZ!

RETARDED POT'S CALLED "LORENTZ POTENTIALS"

LORENTZ (1904, 1909)

TWO ENCYCLOPEDIA ARTICLES, AND HIS BOOK "THEORY OF ELECTRONS"

|| SOLIDIFIED LORENTZ'S POSITION AS A DOMINANT FIGURE AND REFERENCE POINT ||

1904 SOMEWHAT AMBIGUOUS ACCOUNT OF THE LOREN(T)Z CONDITION, $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0$, AND ARBITRARINESS OF CHOICE OF GAUGE (NOT THAT NAME).

1909 FULL DISCUSSION OF $\Phi \to \Phi' = \Phi - \frac{1}{c}\frac{\partial \chi}{\partial t}$, ETC. BUT WITH INTENT ALWAYS TO CHOOSE χ SO THAT $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c}\frac{\partial \Phi'}{\partial t} = 0$, WHICH LORENTZ VIEWED AS NATURAL.

PRE-QUANTUM [WEYL, 1919]

GENERALIZATION OF EINSTEIN'S GR TO INCLUDE ELECTROMAGNETISM (DIDN'T WORK)

WEYL INTRODUCED A LOCAL CHANGE OF SCALE, $g^{\mu\nu} \to g'^{\mu\nu} = g^{\mu\nu} e^{\lambda(x)}$

$\lambda(x)$ REAL; INVARIANCE UNDER THIS TRANSFORMATION, WEYL CALLED "EICHINVARIANZ" (EICHEN = TO MEASURE, CALIBRATE, GAUGE)

HASTY DISCUSSION OF 1926–1929

GAUGE TRANSFORMATION OF WAVE FUNCTION ARISES MOST NATURALLY IN RELATIVISTIC WAVE EQUATION (KNOWN TODAY AS KLEIN-GORDON EQUATION)

$$(p - eA/c)\cdot(p - eA/c) = m^2$$

i.e.

$$\left[(\partial^\mu + ie\,A^\mu/\hbar c)(\partial_\mu + ie\,A_\mu/\hbar c) + (mc/\hbar)^2\right]\psi = 0$$

SCHRODINGER (LATE 1925, JANUARY 1926)

SOLVED K-G COULOMB PROBLEM

DID NOT GET SOMMERFELD FINE-STRUCTURE FORMULA

DISAPPOINTED, DID NOT PUBLISH; WENT NR

(MENTIONED REL. EQN. IN HIS FIRST PAPER)

PAULI (EARLY APRIL, LETTER TO JORDAN)

GETS K-G EQN. WITH STATIC POTENTIAL

DOES NOT SOLVE IT

KLEIN (LATE APRIL)

SUBMITS PAPER ON K-G EQN. IN 5-D FORMALISM. DOES NOT SOLVE, GOT NR LIMIT

FOCK (JUNE 1926)

SUBMITS PAPER DERIVING K-G EQN. FROM VARIATIONAL PRINCIPLE AND SOLVES COULOMB PROBLEM

Erwin Schrödinger
(1887 - 1961)

Oskar Klein
(1894 - 1977)

Vladimir Alexandrovich Fock
(1898 - 1974)

SCHRODINGER (LATE JUNE)

SUBMITS HIS 4TH PAPER IN SERIES; HAS SECTION ON REL. EQN. AND CITES ANSWERS FOR COULOMB PROBLEM AND ZEEMAN EFFECT.

* FOCK (JULY 1926) *

SUBMITS SECOND PAPER ON REL. EQN WITH Φ AND $\vec{A}$. DEMONSTRATES GAUGE TRANSF. INCLUDING

$$\psi \rightarrow \psi' = \psi e^{ie\chi/\hbar c}$$

MENTIONS THAT KLEIN'S "BEAUTIFUL WORK" ARRIVED IN LENINGRAD WHILE HE WAS CORRECTING PROOFS.

GORDON (END OF SEPTEMBER)

SUBMITS PAPER ANALYZING THE COMPTON EFFECT USING REL. WAVE EQN.

DOES NOT CITE KLEIN, FOCK, OR SCHROD. IV

TWO CONCLUSIONS:

(1) FOCK DISCOVERED QUANTUM GAUGE TRANSFORMATION

(2) "KLEIN-GORDON" EQUATION SHOULD BE

(SCHRODINGER)(PAULI) KLEIN-FOCK-SCHRODINGER

EQUATION.

Hermann Weyl

9 November 1885 – 8 December 1955

GAUGE INVARIANCE AS BASIC PRINCIPLE

IN MID-1920s WEYL HAD BEEN DOING MATHEMATICS, NOT PHYSICS, BUT 1927 SAW HIM WRITING HIS "GROUP THEORY AND QUANTUM MECHANICS"

LONDON (EARLY 1927) - SHORT + LONG PAPERS

NOTICED THAT FOCK'S PHASE FACTOR WAS LIKE WEYL'S SCALE FACTOR $\ell^{\lambda(x)}$ EXCEPT FOR A FACTOR i.

LONDON TRIED TO FORCE THINGS INTO A "SCALE" CHANGE BY WRITING

$$\psi/\ell = \psi_0/\ell_0 \text{, WITH } \ell = \ell_0 e^{i\lambda(x)}$$

|| LONDON IDENTIFIED FOCK'S "GRADIENT INVARIANCE" WITH HIS (MODIFIED) "EICHINVARIANZ" OF WEYL. ||

WEYL (1928, 1929)

JUMPED AT LONDON'S TRANSFERENCE OF HIS (FAILED) EICHINVARIANCE IN GR TO QUANTUM MECHANICS

HIS BOOK (1928) - MORE OR LESS FOCK

TWO PAPERS (1929) - ENGLISH, THEN GERMAN

IN THE ENGLISH VERSION, THE TERM "GAUGE INVARIANCE" IS USED FOR THE FIRST TIME.

EVOLUTION OF WEYL'S THINKING

IN HIS BOOK, AFTER DERIVING THE FOCK TRANSFORMATION, WEYL SAYS (IN TRANSLATION)

> "This *'principle of gauge invariance'* is quite analogous to that previously set up by the author, on speculative grounds, in order to arrive at a unified theory of gravitation and electricity[22]. But I now believe that this gauge invariance does not tie together electricity and gravitation, but rather *electricity and matter* in the manner described above."

His note 22 refers to his own work, to Schrödinger (1923), and to London (1927b). In the first (1928) edition, the next sentence reads (again in translation):

> "How gravitation according to general relativity must be incorporated is not certain at present."

IN THE FIRST VERSION OF THE 1929 PAPER (IN ENGLISH, PROC. NAT. ACAD. SC., 1929) HE IS TREATING RELATIVISTIC CHARGED PARTICLES AND EM FIELDS. HE SAYS

> "This new principle of gauge invariance, which may go by the same name, has the character of general relativity since it contains an arbitrary function λ, and can certainly only be understood with reference to it."

IN THE SECOND, SLIGHTLY LONGER (GERMAN) VERSION, WEYL ELABORATED ON THE APPARENT NEED FOR GR:

" In special relativity one must regard this gauge-factor as a constant because here we have only a single point-independent tetrad. Not so in general relativity; every point has its own tetrad and hence its own arbitrary gauge-factor: because by the removal of the rigid connection between tetrads at different points the gauge-factor necessarily becomes an arbitrary function of position." (translation taken from O'Raifeartaigh and Straumann, 2000, p. 7) : (NOT TRUE)

(TRUE)

Nevertheless, Weyl stated (Weyl, 1929a, p.332, below equation (8)),

"If our view is correct, then the electromagnetic field is a necessary accompaniment of the matter wave field and not of gravitation."

The last sentence of (Weyl, 1929b) contains almost the same words. His viewpoint about the need for general relativity can perhaps be understood in the sense that λ *must be* an arbitrary function in the curved space-time of general relativity, but not necessarily in special relativity, and his desire to provide continuity with his earlier work.

WEYL'S PAPERS ARE A WATERSHED

ENSHRINED AS FUNDAMENTAL THE MODERN PRINCIPLE OF GAUGE INVARIANCE, IN WHICH THE EXISTENCE OF THE 4-VECTOR POTENTIALS (AND ASSOCIATED FIELDS) FOLLOW FROM THE REQUIREMENT OF THE INVARIANCE OF THE MATTER EQUATIONS UNDER LOCAL PHASE TRANSFORMATIONS OF THE MATTER FIELDS.

HISTORICAL ROOTS OF GAUGE INVARIANCE

PART 2 (1919 - 1929)

PRE-QUANTUM: WEYL, 1919

GENERALIZATION OF EINSTEIN'S GR TO INCLUDE ELECTROMAGNETISM (DIDN'T WORK)

WEYL'S GENERALIZATION:

LOCAL CHANGE OF SCALE $g^{\mu\nu} \to g'^{\mu\nu} = g^{\mu\nu} e^{\lambda(x)}$

$\lambda(x)$ REAL

POSTULATE: EQUATIONS INVARIANT UNDER THIS TRANSFORMATION

WEYL CALLED THE INVARIANCE,

$\boxed{\text{"EICHINVARIANZ"}}$

(EICHEN = TO MEASURE, CALIBRATE, GAUGE)

WEYL WROTE DIFFERENTIAL FORM FOR CHANGE OF LENGTH SCALE AS

$$dl = l\ \varphi_\nu dx^\nu$$

WITH FORMAL SOLUTION,

$$l = l_0 e^{\lambda(x)}, \quad \lambda(x) = \int^x \varphi_\nu dx^\nu$$

HE WENT ON TO SHOW THAT $F^{\mu\nu}$, DEFINED BY $F^{\mu\nu} = \partial^\mu \varphi^\nu - \partial^\nu \varphi^\mu$ (USUAL DEFINITION), SATISFIED THE FREE MAXWELL EQUATIONS.

FLAW POINTED OUT BY EINSTEIN (AND PAULI)

— IF $\int^x \varphi_\nu \, dx^\nu$ DEPENDS ON PATH, SCALE OF LENGTH AT ANY POINT IS NOT PERMANENTLY DEFINED — PHYSICS FAILS AS A SCIENCE!

SCHRÖDINGER, 1922-23: BOHR + WEYL

E.S. HAD READ WEYL'S "SPACE-TIME-MATTER" (PUBL. 1921) IN WHICH WEYL DESCRIBED HIS 1919 THEORY. HE WAS STRUCK BY THE $e^{\lambda(x)}$ FACTOR AND APPLIED IT TO THE BOHR ATOM: [ZEIT. f. PHYSIK 12, 13-23 (1923)]

Über eine bemerkenswerte Eigenschaft der Quantenbahnen eines einzelnen Elektrons.

Von **Erwin Schrödinger** in Zürich.

(Eingegangen am 5. Oktober 1922.)

"ON A REMARKABLE PROPERTY OF THE QUANTIZED ORBITS OF A SINGLE ELECTRON"

SCHRÖDINGER TOOK $\varphi^\nu = \frac{e}{\gamma} A^\nu$ (γ A CONST.)

(1) SIMPLE ATOM : $\varphi^\nu = \left(\frac{V}{\gamma}, \vec{0}\right)$

$$\oint \varphi_\nu dx^\nu = \oint \frac{V}{\gamma} dt = \frac{\tau}{\gamma} \overline{V}$$

BUT SOMMERFELD-WILSON QUANTIZATION RULE

IS $\oint p\, dq = \oint \vec{p} \cdot \vec{v}\, dt = 2\tau \overline{T} = nh$

VIRIAL THEOREM : $2\overline{T} = -\overline{V}$

$$\therefore \boxed{\oint \varphi_\nu dx^\nu = -\frac{nh}{\gamma}}$$

(2) E.S. GOES ON TO DISCUSS THE ZEEMAN EFFECT STARK EFFECT, AND COMBINED Z. + STARK.

IN HIS DISCUSSION OF RESULTS, HE REMARKS ON THE AMAZING FACTOR OF $e^{h/\gamma}$

WHAT CANBE THE VALUE OF γ ?

(a) DIMENSIONS OF h (ACTION). NATURAL UNIT?

WHAT ABOUT e^2/c ?

WAIT! THEN $\exp\left(\frac{h}{\gamma}\right) \sim e^{2\pi \times 137} \sim e^{1000}$!

UNNATURAL!

(b) MORE LIKELY, $\gamma = O(h)$.

E.S. THEN SAYS ANOTHER POSSIBILITY IS THUS A PURELY IMAGINARY VALUE,

$\boxed{\gamma = \frac{h}{2\pi i}}$ THEN $\exp\left(\frac{h}{\gamma}\right) = e^{2\pi i} = 1$!

END OF PAPER

IN FALL 1925, DE BROGLIE'S THESIS WAS PUBLISHED AND SCHRÖDINGER WAS ASKED TO GIVE A JOURNAL CLUB REPORT ON IT.

ON 3 NOV 1925 HE WROTE EINSTEIN:

A few days ago I read with the greatest interest the ingenious thesis of Louis de Broglie, which I finally got hold of; with it section 8 of your second paper on degeneracy has also become clear to me for the first time. The de Broglie interpretation of the quantum rules seems to me to be related to my note in *Zeit. f. Physik 12, 13, 1922*, where a remarkable property of the Weyl 'gauge factor' along every quasi-period is shown. As far as I can see, the mathematical content is the same, only mine is much more formal, less elegant, and not actually shown in general. Naturally de Broglie's consideration in the framework of his comprehensive theory is altogether of far greater value than my single statement, which I did not know what to make of at first.[3]

IN DECEMBER, SCHRÖDINGER WENT ON A SKIING VACATION WITH A FEMALE FRIEND.

WHILE THERE HE WORKED VIGOROUSLY ON BOTH HIS LOVE LIFE AND HIS PHYSICS (ALL AGREE ON THE MUTUAL STIMULATION)

IT IS CLEAR THAT HIS APPLICATION OF WEYL'S SCALE FACTOR WAS INFLUENTIAL IN SCHRÖDINGER'S DEVELOPMENT OF HIS WAVE EQUATION.

TO DEMONSTRATE THE CLOSE CONNECTION TO SCHRÖDINGER, 1923, LOOK AT THE FORMAL SOLUTION OF THE "K-G" EQUATION,

$$\left[(\partial^{\mu} + ie\, A^{\mu}/\hbar c)(\partial_{\mu} + ie\, A_{\mu}/\hbar c) + (mc/\hbar)^2\right]\psi = 0 \quad ,$$

NAMELY,

$$\boxed{\psi = exp\left(- i\frac{e}{\hbar c}\int^{x} A_{\nu}dx^{\nu}\right)\psi_0}$$

where ψ_0 is the zero-field solution.

THE INTEGRAL PHASE FACTOR IS SCHRÖDINGER'S, IF WE CHOOSE HIS SPECULATIVE $\gamma = -i\hbar$. !

WITH A GAUGE TRANSFORMATION,

$$A^{\mu} \rightarrow A'^{\mu} = A^{\mu} - \partial^{\mu}\chi \quad ,$$

INTEGRATION GIVES (UP TO A CONSTANT PHASE) FOCK'S

$$\psi' = exp(ie\chi(x)/\hbar c)\ \psi \quad ,$$

Schrödinger's first paper in 1926 :

- submitted on 27 January; published on 13 March.
- devoted largely to the nonrelativistic time-independent wave equation problems
- BUT in the second paragraph of Sect. 3, Schrödinger says (in translation):

> "For example, if the relativistic Kepler problem is worked out, it is found to lead in a remarkable manner to *half-integral partial quanta* (radial and azimuthal)."

- obviously S. had written down and solved the relativistic wave equation with a static Coulomb potential, but had not obtained the Sommerfeld fine-structure formula.

What is this business of getting "half-integral partial quanta" ?

Sommerfeld's fine-structure formula (formally the same as the Dirac result, still to come) is

$$E = mc^2 \left[1 + \frac{(\alpha Z)^2}{\lambda_S^2} \right]^{-1/2}$$

where

$$\lambda_S = n' + \sqrt{k^2 - (\alpha Z)^2} \quad ; \quad k = \pm 1, \pm 2, \pm 3, \ldots .$$

Note that $k^2 = (j + 1/2)^2$, and $n' = 0, 1, 2, \ldots\ldots$ is the radial quantum number. Note that in λ_s both k and n' are integers.

What is the answer for the Klein-Gordon Coulomb problem?

It has the same form as the Sommerfeld-Dirac result, but with λ_S replaced by λ_{KG}:

$$\lambda_{KG} = n' + 1/2 + \sqrt{(\ell + 1/2)^2 - (\alpha Z)^2} \quad ; \quad \ell = 0, 1, 2, \ldots .$$

λ_{KG} differs from λ_S by having $n' \rightarrow n' + 1/2$ and $k \rightarrow \ell + 1/2$. That is why Schrödinger spoke of "half-integral partial quanta" and why he was disappointed.

DECEMBER 7, 1926 LONDON → SCHRODINGER

In 1926, Fritz London wrote a remarkably playful letter to Schrödinger about this paper:[6]

Very Respected Herr Professor:
Today I must talk with you seriously. Do you know a certain Herr Schrödinger who described, in the year 1922, a 'noteworthy property of quantum orbits'? Do you know this man? What, you say you know him rather well, you were even with him when he wrote this paper and were implicated in the work? This is truly shocking. Hence you already knew four years ago that one does not possess rods and clocks for the definition of an Einstein–Riemannian measure in the continuous description that occurs in analyzing atomic processes; thus one must see whether perhaps the general principles of measurement that arise from Weyl's theory of distance transfer might help. And you even realized four years ago that thev help very well . . . and you showed that for real discrete orbits the gauge factor reproduces itself on a spatially closed path; and especially you then realized that on the nth orbit the unit of measure swells and shrinks n times, exactly as in the case of a standing wave describing the position of charge. You therefore demonstrated that Weyl's theory becomes reasonable – i.e., *leads to a unique determination of measure* – only if combined with quantum theory; and one has no other choice if the whole world of atoms represents a process in a continuum without any identifiable fixed point. You knew this and said nothing about it . . . Will you now immediately confess that, like a priest, you held the truth in your hands and kept it a secret? . . .